Name:____________________________

Student ID:____________________________

Section:____________________________

Instructor:____________________________

Math 113 (Calculus II)
Midterm Exam 3
Solutions

March 26–March 30, 2010

Instructions:

- Work on scratch paper will not be graded.
- For questions 12 to 17, show **all** your work in the space provided. Full credit will be given only if the necessary work is shown justifying your answer. Please write neatly.
- Should you have need for more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.
- Simplify your answers. Expressions such as ln(1), e^0, $\sin(\pi/2)$, etc. must be simplified for full credit.
- Calculators are not allowed.

For Instructor use only.

#	Possible	Earned
MC	36	
10	6	
11	5	
12a	5	
12b	5	
13a	5	
13b	5	
Sub	67	

#	Possible	Earned
13c	5	
14	6	
15a	5	
15b	5	
16	6	
17	6	
Sub	33	
Total	100	

Multiple Choice. Fill in the answer to each problem on your computer-scored answer sheet. Make sure your name, section and instructor are on that sheet.

1. Which of the following is true for $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$?

   a) The series converges conditionally.
   b) The series diverges.
   c) The series converges absolutely.
   d) None of the above.

   ANSWER: A

2. Find the arclength of the curve $x = e^t, y = 2$ for $0 \leq t \leq 1$.

   a) $e - 1$
   b) e
   c) Impossible to evaluate
   d) $\sqrt{e}$
   e) $1 - \sqrt{e}$
   f) None of the above.

   ANSWER: A

3. Which test for convergence would be appropriate if you wanted to test $\sum_{n=1}^{\infty} \frac{2^n}{n!}$?

   a) Integral Test.
   b) Alternating Series Test
   c) Ratio Test
   d) Comparison Test
   e) Limit Comparison Test
   f) Root Test.
   g) None of the above.

   ANSWER: C

4. The radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{\sqrt{n^3}} (x-1)^n$$

   is

   a) 0
   b) $\frac{1}{3}$
   c) 1
   d) 3
   e) ∞
   f) None of these.

   ANSWER: B

5. What is the sum of the series

$$1 - \ln 3 + \frac{(\ln 3)^2}{2} - \frac{(\ln 3)^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{(-\ln 3)^n}{n!}?$$

a) 0 b) e c) $\frac{1}{3}$

d) $\frac{1}{e}$ e) 3 f) None of these.

ANSWER: C

6. Which of the following is true about the series $\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

a) The series diverges by the Ratio Test because the limit ratio is 2.

b) The Ratio Test gives no information.

c) The series converges by the Ratio Test because the limit ratio is $\frac{1}{2}$.

d) The series converges by the Ratio Test because the limit ratio is $\frac{1}{3}$.

e) The series converges by the Ratio Test because the limit ratio is 0.

ANSWER: C

7. Which of the following equations can be obtained by eliminating the parameters from

$$x = t^2 - 2, y = 5 - 2t?$$

a) $x = 5 - 2\sqrt{y+2}$ b) $x = \left(\frac{5-y}{2}\right)^2 - 2$

c) $x = 23 - 20y + 4y^2$ d) $x = 9 - y^2$

e) $x = y^2 - 2y + 3$

ANSWER: B

8. What is the coeffieient of x^4 in the MacLaurin series for $e^x \cos x^2$

a) $\frac{13}{24}$ b) $-\frac{13}{24}$ c) $\frac{11}{24}$

d) $-\frac{11}{24}$ e) $\frac{15}{24}$ f) None of the above.

ANSWER: D

9. At what (x, y) values does the curve $x = 10 - 4t^2, y = 8t^3 - 24t$ have a vertical tangent?

a) $(6, -16), (6, 16)$ b) $(0, 0)$

c) $(\frac{\sqrt{10}}{2}, -11\sqrt{10})$ d) $(\frac{\sqrt{10}}{2}, -11\sqrt{10}), (\frac{-\sqrt{10}}{2}, 11\sqrt{10})$

e) $(10, 0)$ f) $(10, 0), (6, 16), (6, -16)$

g) None of the above.

ANSWER: E

Short Answer: Write your answer in the space provided. Answers not placed in this space will be ignored.

10. (6 points) Answer the following:

(a) The Maclaurin series for $\sin(2x)$ is: ____________________

SOLUTION:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

$$\text{so } \sin(2x) = (2x) - \frac{(2x)^3}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!}.$$

(b) The curves

$$y = 2 + 3\cos t, x = 3\sin t, 0 \le t \le 2\pi$$

parametrize a circle of radius ____________________

centered at ____________________

.

SOLUTION: Radius = 3, center at $(0, 2)$.

11. (5 points) Evaluate the following limit: $\lim_{x\to 0} \dfrac{3\tan^{-1} x^2 - 3x^2 + x^6}{x^{10}}$. ____________________

SOLUTION:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + \cdots \\
\frac{1}{1+x^2} &= 1 - x^2 + x^4 - x^6 + \cdots \\
\tan^{-1}(x) &= \int \frac{1}{1+x^2}\, dx \\
&= \int 1 - x^2 + x^4 - x^6 + \cdots \\
&= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} \\
3\tan^{-1} x^2 &= 3\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} [(x^2)^{2n+1}] \\
&= 3x^2 - x^6 + \frac{3}{5}x^{10} - \frac{3}{7}x^{14} + \cdots \\
\lim_{n\to\infty} \frac{3\tan^{-1} x^2 - 3x^2 + x^6}{x^{10}} &= \lim_{n\to\infty} \frac{3}{5} - \frac{3}{7}x^4 + \cdots = \frac{3}{5}
\end{aligned}$$

Free response: Write your solution and answer in the space provided. Answers not placed in this space will be ignored.

12. Determine whether the following series converge, and justify your answer.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3^n}$

SOLUTION: Notice that $\displaystyle\frac{1}{n+3^n} < \frac{1}{3^n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges since it is a geometric series with $r = \dfrac{1}{3} < 1$. So by the comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3^n}$ converges.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} n \sin(\frac{1}{n})$

SOLUTION: $\displaystyle\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \neq 0$, so $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ diverges by the divergence test.

13. Find the interval of convergence for the following series:

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 2^n}(x+1)^n$

SOLUTION: Using the ratio test we have:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(n+1)^2 2^{n+1}}\cdot\frac{n^2 2^n}{(x+1)^n}\right| &= \lim_{n\to\infty}\frac{|x+1|}{2}\left(\frac{n}{n+1}\right)^2 \\ &= \frac{|x+1|}{2}\end{aligned}$$

This converges when $\dfrac{|x+1|}{2} < 1 \Longrightarrow |x+1| < 2 \Longrightarrow -2 < x+1 < 2 \Longrightarrow -3 < x < 1$.
Now we need to check the endpoints.
At $x = -3$, we have $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 2^n}(-2)^n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ which converges by the p-test.
At $x = 1$, we have $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 2^n}2^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ which converges by the alternating series test.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^4}x^n$

SOLUTION: Using the ratio test we have:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(n+1)^4}\cdot\frac{n^4}{n!x^n}\right| &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^4(n+1)|x| \\ &= \infty \text{ if } x\neq 0 \text{ and } = 0 \text{ if } x = 0\end{aligned}$$

Thus the interval of convergence is $\{0\}$.

(c) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n(n-1)}{n^2+2}(x-3)^{2n}$

SOLUTION:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{2^{n+1}(n)(x-3)^{2(n+1)}}{(n+1)^2+2}\cdot\frac{n^2+2}{2^n(n-1)(x-3)^{2n}}\right| &= \lim_{n\to\infty} 2\left(\frac{n}{n-1}\right)\left(\frac{n^2+2}{(n+1)^2+2}\right)|x-3|^2 \\
&= 2|x-3|^2
\end{aligned}$$

Thus the series converges if $2|x-3|^2 < 1 \implies |x-3|^2 < \frac{1}{2} \implies \frac{-1}{\sqrt{2}} < x-3 < \frac{1}{\sqrt{2}} \implies 3-\frac{1}{\sqrt{2}} < x < 3+\frac{1}{\sqrt{2}}$.

Check the endpoints:

At $x = 3-\frac{1}{\sqrt{2}}$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n(n-1)}{n^2+2}\left(\frac{-1}{\sqrt{2}}\right)^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^n(n-1)}{n^2+2}$ which converges by the alternating series test.

At $x = 3+\frac{1}{\sqrt{2}}$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n(n-1)}{n^2+2}\left(\frac{1}{\sqrt{2}}\right)^{2n} = \sum_{n=1}^{\infty}\frac{(-1)^n(n-1)}{n^2+2}$ which converges by the alternating series test.

Thus the interval of convergence is $\left[3-\frac{1}{\sqrt{2}}, 3+\frac{1}{\sqrt{2}}\right]$.

14. (6 points) Find the first four nonzero terms of a power series for

$$f(x) = (27-x)^{1/3}.$$

SOLUTION: We use the binomial theorem to get the following:

$$\begin{aligned}
f(x) &= (27-x)^{1/3} \\
&= [27(1-\frac{x}{27})]^{1/3} \\
&= 3\left[1+\left(\frac{-x}{27}\right)\right]^{1/3} \\
&= 3\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}x^n \\
&= 3\left[1+\frac{1}{3}\left(\frac{-x}{27}\right)+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2!}\left(\frac{-x}{27}\right)^2+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{3!}\left(\frac{-x}{27}\right)^3+\cdots\right] \\
&= 3\left[1-\frac{x}{81}-\frac{1}{6561}x^2-\frac{5}{1594323}x^3-\cdots\right] \\
&= 3-\frac{x}{27}-\frac{x^2}{3\cdot 27^2}-\frac{5x^3}{27^4}-\cdots
\end{aligned}$$

15. (a) (5 points) Find a power series representation for

$$\int_0^1 \frac{x^2}{1+x^6}\,dx.$$

SOLUTION:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \\
\frac{1}{1+x^6} &= 1 - x^6 + x^{12} - x^{18} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{6n} \\
\frac{x^2}{1+x^6} &= x^2 - x^8 + x^{14} - x^{20} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{6n+2} \\
\int_0^1 \frac{x^2}{1+x^6}\,dx &= \int_0^1 (x^2 - x^8 + x^{14} - x^{20} + \cdots)\,dx = \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{6n+2}\,dx \\
&= \frac{1}{3}x^3 - \frac{1}{9}x^9 + \frac{1}{15}x^{15} - \frac{1}{21}x^{21} + \cdots |_0^1 \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{6n+3} |_0^1 \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{6n+3}
\end{aligned}$$

(b) (5 points) Suppose we approximate the integral in part (a) by summing the first k terms of the series. How large does k need to be so that the error of our approximation is no greater than 0.0006?

SOLUTION: We want the following to hold:

$$\begin{aligned}
\frac{1}{6n+3} &< 0.0006 = \frac{6}{10,000} \\
10,000 &< 36n + 18 \\
9982 &< 36n \\
277 &< n
\end{aligned}$$

Thus $k = 278$ since there are $k = n + 1$ terms in the summation.

16. (6 points) Find an equation for the tangent line to the curve at the point corresponding to $t = 1$ on $x = t^4 + 1, y = t^3 + t$.

SOLUTION:

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{3t^2+1}{4t^3} \\ \frac{dy}{dx}\Big|_{t=1} &= \frac{4}{4} \\ &= 1 \end{aligned}$$

Thus we want the equation of the line with slope $m = 1$ through the point $(2, 2)$, which is:

$$\begin{aligned} y - 2 &= 1(x - 2) \\ y &= x \end{aligned}$$

17. (6 points) Find the surface area generated by rotating the given curve about the y-axis:

$$x = e^t - t, y = 4e^{t/2}, 0 \leq t \leq 1$$

SOLUTION: Notice that the radius of the surface of rotation is given by the x-value, so $r = e^t - t$. Also, $\frac{dx}{dt} = e^t - 1$, and $\frac{dy}{dt} = 2e^{t/2}$, so we have:

$$
\begin{aligned}
\text{Surface Area} &= \int_0^1 2\pi(e^t - t)\sqrt{(e^t-1)^2 + (2e^{t/2})^2}\,dt \\
&= 2\pi \int_0^1 (e^t - t)\sqrt{e^{2t} - 2e^t + 1 + 4e^t}\,dt \\
&= 2\pi \int_0^1 (e^t - t)\sqrt{e^{2t} + 2e^t + 1}\,dt \\
&= 2\pi \int_0^1 (e^t - t)\sqrt{(e^t + 1)^2}\,dt \\
&= 2\pi \int_0^1 (e^t - t)(e^t + 1)\,dt \\
&= 2\pi \int_0^1 (e^{2t} - te^t + e^t - t)\,dt \text{(use integration by parts)} \\
&= 2\pi \left[\frac{1}{2}e^{2t} - te^t + e^t + e^t - \frac{1}{2}t^2\right]_0^1 \\
&= 2\pi \left[\frac{1}{2}e^{2t} - te^t + 2e^t - \frac{1}{2}t^2\right]_0^1 \\
&= 2\pi \left[\left(\frac{1}{2}e^2 - e + 2e - \frac{1}{2}\right) - \left(\frac{1}{2} + 2\right)\right] \\
&= 2\pi \left[\frac{1}{2}e^2 + e - 3\right] \\
&= \pi\left[e^2 + 2e - 6\right]
\end{aligned}
$$

END OF EXAM